

A CRITERION FOR THE PROPERNESS OF THE K -ENERGY IN A GENERAL KÄHLER CLASS (II)

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ABSTRACT. In this paper, we give a result on the properness of the K -energy, which answers a question of Song-Weinkove [11] in any dimensions. Moreover, we extend our previous result on the properness of K -energy in [9] to the case of modified K -energy associated to extremal Kähler metrics.

CONTENTS

1. Introduction	1
2. Proof of Theorem 1.1	3
3. Proof of Theorem 1.2	4
3.1. The modified K -energy $\tilde{\mu}$ and \tilde{J} functional	4
3.2. The existence of critical points of \tilde{J}	6
3.3. Proof of Theorem 1.2	11
References	13

1. INTRODUCTION

This paper is a continuation of our previous work [9]. In [9], we give a criterion for the properness of the K -energy in a general Kähler class of a compact Kähler manifold by using Song-Weinkove's result on J -flow in [11], which extends the works of Chen [1], Song-Weinkove [11] and Fang-Lai-Song-Weinkove [6]. In [11], Song-Weinkove showed that the K -energy is proper on a Kähler class $[\chi_0]$ of a n -dimensional Kähler manifold M with $c_1(M) < 0$ whenever there are Kähler metrics $\omega \in -\pi c_1(M)$ and $\chi' \in [\chi_0]$ such that

$$\left(-n \frac{\pi c_1(M) \cdot [\chi_0]^{n-1}}{[\chi_0]^n} \chi' - (n-1)\omega \right) \wedge \chi'^{n-2} > 0. \quad (1.1)$$

Moreover, Song-Weinkove asked whether the K -energy is bounded from below if the inequality (1.1) is not strict (Remark 4.2 of [11]). In [6] Fang-Lai-Song-Weinkove studied the J -flow on the boundary of the Kähler cone and gave an affirmative answer in complex dimension 2. In [9], we give a partial answer to this question, which says that the K -energy is proper if

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$c_1(M) < 0$ and the Kähler class $[\chi_0]$ satisfies

$$-n \frac{c_1(M) \cdot [\chi_0]^{n-1}}{[\chi_0]^n} [\chi_0] + (n-1)c_1(M) \geq 0.$$

The first main result in this paper is the following theorem, which answers the question of Song-Weinkove in any dimensions.

Theorem 1.1. *Let M be a n -dimensional compact Kähler manifold with $c_1(M) < 0$. If the Kähler class $[\chi_0]$ satisfies the property that there are two Kähler metrics $\chi' \in [\chi_0]$ and $\omega \in -\pi c_1(M)$ such that*

$$\left(-n \frac{\pi c_1(M) \cdot [\chi_0]^{n-1}}{[\chi_0]^n} \chi' - (n-1)\omega \right) \wedge \chi'^{n-2} \geq 0, \quad (1.2)$$

then the K -energy is proper on the Kähler class $[\chi_0]$.

Our second main result is to extend [9] to the case of extremal Kähler metrics. To state our main results, we recall Tian's α -invariant for a Kähler class $[\chi_0]$:

$$\alpha_M([\chi_0]) = \sup \left\{ \alpha > 0 \mid \exists C > 0, \int_M e^{-\alpha(\varphi - \sup \varphi)} \chi_0^n \leq C, \quad \forall \varphi \in \mathcal{H}(M, \chi_0) \right\},$$

where $\mathcal{H}(M, \chi_0)$ denotes the space of Kähler potentials with respect to the metric χ_0 . For any compact subgroup G of $\text{Aut}(M)$, and a G -invariant Kähler class $[\chi_0]$, we can similarly define the $\alpha_{M,G}$ invariant by using G -invariant potentials in the definition.

Theorem 1.2. *Let M be a n -dimensional compact Kähler manifold and X an extremal vector field of the Kähler class $[\chi_0]$ with potential function $\theta_X := \theta_X(\chi_0)$. Assume $\text{Im} X$ generates a compact group of holomorphic automorphisms¹, and $L_{\text{Im} X} \chi_0 = 0$. If the Kähler class $[\chi_0]$ satisfies the following conditions for some constant ϵ :*

- (1) $0 \leq \epsilon < \frac{n+1}{n} \alpha_M([\chi_0])$,
- (2) $\pi c_1(M) < (\epsilon + \min \theta_X) [\chi_0]$,
- (3)

$$\left(-n \frac{\pi c_1(M) \cdot [\chi_0]^{n-1}}{[\chi_0]^n} + \min_M \theta_X + \epsilon \right) [\chi_0] + (n-1)\pi c_1(M) > 0,$$

then the modified K -energy is proper on $\mathcal{H}_X(M, \chi_0)$, where $\mathcal{H}_X(M, \chi_0)$ is the subspace of $\mathcal{H}(M, \chi_0)$ with the extra condition $\text{Im} X(\varphi) = 0$. If instead of (1), we assume $[\chi_0]$ is G -invariant for a compact subgroup G of $\text{Aut}(M)$, and $0 \leq \epsilon < \frac{n+1}{n} \alpha_{M,G}([\chi_0])$, then the modified K -energy is proper on the space of G -invariant potentials.

For the definitions of extremal vector field and modified K -energy, see section 3. Note that the $\min \theta_X$ here is actually an invariant of the Kähler class according to [10] and Appendix of [17]. We can also replace the condition (3) of Theorem 1.2 by some weaker assumptions as in Theorem 1.1, however we prefer this version since (3) is easier to check. The proof of Theorem 1.2 relies on the study of the modified J -flow, which is an extremal version of the usual J flow defined by Donaldson [5] and Chen [1]. Here we modify the proof of Song-Weinkove [11]

¹If $[\chi_0] = c_1(M)$, then this is always true, see Theorem F of [8].

to get the existence of critical metrics of the modified J functional and then we apply the argument in [9] to get the properness of the modified K -energy.

In a recent interesting paper [4], Dervan gives a different sufficient condition on the properness of the K energy on a general Kähler class by direct analyzing the expression of the K energy, which gives better results in some examples (cf. [9][4]). While Dervan's condition is useful mainly when M is Fano, our theorem applies on more general manifolds. Whether one can improve both results is still an interesting problem.

In section 2, we prove Theorem 1.1, which is a strengthen of the Main Theorem of [9]. Then in section 3, we study the modified J -flow and prove Theorem 1.2.

2. PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1. Here we use the notations in our previous work [9].

Proof of Theorem 1.1. By the assumption (1.2), for sufficiently small $\epsilon > 0$ we have

$$\left((nc + \epsilon)\chi' - (n - 1)\omega\right) \wedge \chi'^{n-2} > 0, \quad (2.1)$$

where

$$c = \frac{-\pi c_1(M) \cdot [\chi_0]^{n-1}}{[\chi_0]^n}.$$

We can write (2.1) as

$$\left(n(c + \epsilon)\chi' - (n - 1)(\omega + \epsilon\chi')\right) \wedge \chi'^{n-2} > 0, \quad (2.2)$$

Since ω and χ' are Kähler metrics, we have $\omega + \epsilon\chi' > 0$. by Song-Weinkove's result (cf. Theorem 1.1 in [11]) there exists a Kähler metric $\chi \in [\chi_0]$ such that

$$(\omega + \epsilon\chi') \wedge \chi^{n-1} = (c + \epsilon)\chi^n.$$

Thus, the functional $\hat{J}_{\omega + \epsilon\chi', \chi_0}$ is bounded from below on $[\chi_0]$. Since $\chi' \in [\chi_0]$, by the argument of [13]² [9] there is a uniform constant $C > 0$ such that for any $\varphi \in \mathcal{H}(M, \chi_0)$,

$$|\hat{J}_{\omega + \epsilon\chi', \chi_0}(\varphi) - \hat{J}_{\omega + \epsilon\chi_0, \chi_0}(\varphi)| \leq C.$$

Therefore, $\hat{J}_{\omega + \epsilon\chi_0, \chi_0}(\varphi)$ is bounded from below and we have

$$\hat{J}_{\omega, \chi_0}(\varphi) \geq -\epsilon \left(I_{\chi_0}(\varphi) - J_{\chi_0}(\varphi) \right) - C, \quad \forall \varphi \in \mathcal{H}(M, \chi_0). \quad (2.3)$$

Now using Tian's α -invariant we have (see Lemma 4.1 of [11], also [14] page 95)

$$\begin{aligned} \int_X \log \frac{\chi_\varphi^n}{\chi_0^n} \frac{\chi_\varphi^n}{n!} &\geq \alpha I_{\chi_0}(\varphi) - C \\ &\geq \frac{n+1}{n} \alpha \cdot (I_{\chi_0}(\varphi) - J_{\chi_0}(\varphi)) - C, \quad \forall \varphi \in \mathcal{H}(M, \chi_0) \end{aligned} \quad (2.4)$$

²The authors would like to thank G. Székelyhidi for telling them this fact, which they overlooked when preparing [9].

for any $\alpha \in (0, \alpha_M([\chi_0]))$. Set $\omega_0 := -Ric(\chi_0) > 0$. Combining the inequalities (2.3)-(2.4) we have

$$\begin{aligned} \mu_{\chi_0}(\varphi) &= \int_X \log \frac{\chi_\varphi^n}{\chi_0^n} \frac{\chi_\varphi^n}{n!} + \hat{J}_{\omega_0, \chi_0}(\varphi) \\ &\geq \int_X \log \frac{\chi_\varphi^n}{\chi_0^n} \frac{\chi_\varphi^n}{n!} + \hat{J}_{\omega, \chi_0}(\varphi) - C \\ &\geq \left(\frac{n+1}{n} \alpha - \epsilon \right) \left(I_{\chi_0}(\varphi) - J_{\chi_0}(\varphi) \right) - C. \end{aligned}$$

Therefore, for sufficiently small ϵ the K energy is proper. \square

3. PROOF OF THEOREM 1.2

3.1. The modified K -energy $\tilde{\mu}$ and \tilde{J} functional. We first recall some notations in [9]. Let (M, χ_0) be a n -dimensional compact Kähler manifold with a Kähler form

$$\chi_0 = \frac{\sqrt{-1}}{2} h_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

We denote by $\mathcal{H}(M, \chi_0)$ the space of Kähler potentials

$$\mathcal{H}(M, \chi_0) = \{ \varphi \in C^\infty(M, \mathbb{R}) \mid \chi_\varphi = \chi_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \varphi > 0 \}.$$

A metric χ is called “extremal” if the gradient of the scalar curvature $R(\chi)$ is a holomorphic vector field, i.e.

$$R(\chi) - \underline{R} - \theta_X(\chi) = 0,$$

where \underline{R} is the integral mean value of $R(\chi)$ (which is a topological number) and $\theta_X(\chi)$ is the normalized holomorphic potential of a holomorphic vector field X with respect to the metric χ . Namely, $\theta_X(\chi)$ satisfies the equalities

$$L_X \chi = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \theta_X(\chi), \quad \int_M \theta_X(\chi) \frac{\chi^n}{n!} = 0.$$

Such a holomorphic vector field X is called an “extremal vector field”. Futaki and Mabuchi proved that “extremal vector field” makes sense in a general Kähler manifold and is unique [8]. Here we always assume that $L_{\text{Im} X} \chi = 0$, hence θ_X is real-valued. For such an extremal vector field X , we modified the space of Kähler potentials accordingly:

$$\mathcal{H}_X(M, \chi_0) = \{ \varphi \in \mathcal{H}(M, \chi_0) \mid \text{Im} X(\varphi) = 0 \}.$$

For any $\varphi \in \mathcal{H}_X(M, \chi_0)$, the potential $\theta_X(\chi_\varphi)$ is also real-valued (Since we always have $\theta_X(\chi_\varphi) = \theta_X(\chi) + X(\varphi)$). Then we can define the modified K -energy on $\mathcal{H}_X(M, \chi_0)$ by the variational formula

$$\delta \tilde{\mu}_{\chi_0}(\varphi) = - \int_M \delta \varphi (R(\chi_\varphi) - \underline{R} - \theta_X(\chi_\varphi)) \frac{\chi_\varphi^n}{n!}.$$

Then the critical point of $\tilde{\mu}_{\chi_0}$ is just an extremal Kähler metric in $[\chi_0]$.

The \tilde{J} functional with respect to a reference closed (1,1)-form ω (not necessarily positive) is defined by the formula

$$\begin{aligned}\tilde{J}_{\omega, \chi_0}(\varphi) &= \hat{J}_{\omega, \chi_0}(\varphi) + \int_0^1 \int_M \frac{\partial \varphi_t}{\partial t} \theta_X(\varphi_t) \frac{\chi_{\varphi_t}^n}{n!} dt \\ &= \int_0^1 \int_M \frac{\partial \varphi_t}{\partial t} (\omega \wedge \chi_{\varphi_t}^{n-1} - c \chi_{\varphi_t}^n) \frac{dt}{(n-1)!} + \int_0^1 \int_M \frac{\partial \varphi_t}{\partial t} \theta_X(\varphi_t) \frac{\chi_{\varphi_t}^n}{n!} dt, \quad (3.1)\end{aligned}$$

where

$$c = \frac{[\omega][\chi_0]^{n-1}}{[\chi_0]^n}.$$

When we choose $\omega_0 = -Ric(\chi_0)$, then a direct computation shows that

$$\tilde{\mu}_{\chi_0}(\varphi) = \int_M \log \frac{\chi_{\varphi}^n}{\chi_0^n} \frac{\chi_{\varphi}^n}{n!} + \tilde{J}_{\omega_0, \chi_0}(\varphi). \quad (3.2)$$

Note that the modified K-energy and J -functional actually make sense on the larger space $\mathcal{H}(M, \chi_0)$, though the value may be not real. However the \tilde{J} functional enjoys the following interesting property as the usual J functional:

Proposition 3.1. *When ω is positive, the real part of $\tilde{J}_{\omega, \chi_0}$ is strictly convex along any $C^{1,1}$ geodesics in $\mathcal{H}(M, \chi_0)$, and its imaginary part is linear.*

Proof. Since the usual \hat{J} functional is real valued and strictly convex along any $C^{1,1}$ geodesics by the work of Chen, we only need to compute the second order derivative of the additional term. Suppose the geodesic is C^2 , then

$$\begin{aligned}\frac{d}{dt} \int_M \dot{\varphi}_t \theta_X(\varphi_t) \frac{\chi_{\varphi_t}^n}{n!} &= \frac{d}{dt} \int_M \dot{\varphi}_t (\theta_X(\chi_0) + X(\varphi_t)) \frac{\chi_{\varphi_t}^n}{n!} \\ &= \int_M \left[\ddot{\varphi}_t \theta_X(\varphi_t) + X\left(\frac{1}{2}\dot{\varphi}^2\right) \right] \frac{\chi_{\varphi_t}^n}{n!} + \int_M \dot{\varphi}_t \theta_X(\varphi_t) \frac{\sqrt{-1}}{2} \partial \bar{\partial} \dot{\varphi}_t \wedge \frac{\chi_{\varphi_t}^{n-1}}{(n-1)!} \\ &= \int_M \left[\ddot{\varphi}_t - \langle \partial \dot{\varphi}_t, \partial \dot{\varphi}_t \rangle \right] \theta_X(\varphi_t) \frac{\chi_{\varphi_t}^n}{n!} + \int_M X\left(\frac{1}{2}\dot{\varphi}^2\right) \frac{\chi_{\varphi_t}^n}{n!} \\ &\quad - \int_M \frac{\sqrt{-1}}{2} \partial \theta_X(\varphi_t) \wedge \bar{\partial} \left(\frac{1}{2}\dot{\varphi}_t^2\right) \wedge \frac{\chi_{\varphi_t}^{n-1}}{(n-1)!} \\ &= \int_M \left[\ddot{\varphi}_t - \langle \partial \dot{\varphi}_t, \partial \dot{\varphi}_t \rangle \right] \theta_X(\varphi_t) \frac{\chi_{\varphi_t}^n}{n!} + \int_M L_X \left(\frac{1}{2} \dot{\varphi}_t^2 \frac{\chi_{\varphi_t}^n}{n!} \right) \\ &= \int_M \left[\ddot{\varphi}_t - \langle \partial \dot{\varphi}_t, \partial \dot{\varphi}_t \rangle \right] \theta_X(\varphi_t) \frac{\chi_{\varphi_t}^n}{n!} = 0.\end{aligned}$$

Then we approximate a general $C^{1,1}$ geodesic by C^2 geodesics as Chen-Tian [3]. So we conclude that $\text{Re} \tilde{J}$ is also strictly convex along any $C^{1,1}$ geodesics, and $\text{Im} \tilde{J}$ is linear. \square

A direct corollary of Proposition 3.1 is the following result:

Corollary 3.2. *If $\tilde{J}_{\omega, \chi_0}$ has a critical point $\varphi \in \mathcal{H}_X(M, \chi_0)$ and $\omega > 0$, then $\tilde{J}_{\omega, \chi_0}$ is bounded from below on $\mathcal{H}_X(M, \chi_0)$.*

Proof. This is a minor modification of Chen's proof of Proposition 3 in [1]. We just connect any $\psi \in \mathcal{H}_X(M, \chi_0)$ with φ by a $C^{1,1}$ geodesic in $\mathcal{H}(M, \chi_0)$. Then since both $\tilde{J}_{\omega, \chi_0}(\varphi)$ and $\tilde{J}_{\omega, \chi_0}(\psi)$ are real valued, by Proposition 3.1, $\tilde{J}_{\omega, \chi_0}$ is real valued along this geodesic, and hence strictly convex. The rest of the proof is identical to that of Chen in [1], so we omit it. \square

3.2. The existence of critical points of \tilde{J} . In this subsection, we always assume ω is a closed positive (1,1)-form, i.e. a Kähler form. We want to find out the critical point of $\tilde{J}_{\omega, \chi_0}$. By definition, a critical point $\varphi \in \mathcal{H}_X(M, \chi_0)$ satisfies

$$\omega \wedge \chi_\varphi^{n-1} = \left(c + \frac{1}{n} \theta_X(\chi_\varphi)\right) \chi_\varphi^n. \quad (3.3)$$

We have a similar theorem as Song-Weinkove, saying that the existence of a “subsolution” (in a suitable sense) to the above Euler-Lagrange equation will actually leads to a solution:

Theorem 3.3. *If there is a metric $\chi' \in [\chi_0]$ satisfying*

$$(nc\chi' - (n-1)\omega) \wedge \chi'^{n-2} + \theta_X(\chi') \chi'^{n-1} > 0, \quad (3.4)$$

and $L_{\text{Im } X} \omega = 0$, then there is a smooth Kähler metric $\chi_\varphi = \chi_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \varphi \in [\chi_0]$ satisfying the equation (3.3), and the solution χ_φ is unique.

Note that (3.4) automatically implies $\theta_X(\chi')$ is real valued. The uniqueness part of Theorem 3.3 follows directly from Proposition 3.1. We only need to study the existence problem. Without loss of generality, we may assume that the initial metric χ_0 satisfies (3.4). To prove Theorem 3.3, we introduce the following flow, called “modified J-flow”:

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= c - \frac{\omega \wedge \chi_\varphi^{n-1}}{\chi_\varphi^n} + \frac{1}{n} \text{Re } \theta_X(\chi_\varphi) \\ &= \frac{1}{n} \left(nc + \text{Re } \theta_X(\chi_\varphi) - \Lambda_{\chi_\varphi} \omega \right). \end{aligned} \quad (3.5)$$

Denote the right hand side operator by $L(\varphi)$, then it is easy to see that the linearization of L is given by $\tilde{\Delta} + \frac{1}{n} \text{Re } X$, where

$$\tilde{\Delta} f = \frac{1}{n} h^{k\bar{l}} \partial_k \partial_{\bar{l}} f, \quad h^{k\bar{l}} = \chi^{k\bar{j}} \chi^{i\bar{l}} g_{i\bar{j}}.$$

Since $\tilde{\Delta}$ is strictly elliptic, we always have short time solution to the flow equation (3.5).

A modified J -flow starts with an element of $\mathcal{H}_X(M, \chi_0)$ will remain in this space:

Lemma 3.4. *If $\varphi_0 = \varphi|_{t=0}$ satisfies $(\text{Im } X)(\varphi_0) = 0$, and $L_{\text{Im } X} \omega = 0$, then along the modified J -flow, we always have $(\text{Im } X)(\varphi) = 0$.*

Proof. Denote $\text{Im } X$ by Y . By the assumption $L_Y \omega = 0 = L_Y \chi_0$, we have

$$\begin{aligned} Y(\Lambda_{\chi_\varphi} \omega) &= -\chi_\varphi^{\alpha\bar{j}} \chi_\varphi^{i\bar{\beta}} (L_Y \chi_\varphi)_{\alpha\bar{\beta}} g_{i\bar{j}} + \chi_\varphi^{i\bar{j}} (L_Y \omega)_{i\bar{j}} \\ &= -\chi_\varphi^{\alpha\bar{j}} \chi_\varphi^{i\bar{\beta}} (L_Y \chi_0 + Y(\varphi))_{\alpha\bar{\beta}} g_{i\bar{j}} \\ &= -\chi_\varphi^{\alpha\bar{j}} \chi_\varphi^{i\bar{\beta}} (Y(\varphi))_{\alpha\bar{\beta}} g_{i\bar{j}} = -n \tilde{\Delta} Y(\varphi) \end{aligned}$$

Form (3.5), we have

$$\begin{aligned}\frac{\partial Y(\varphi)}{\partial t} &= -\frac{1}{n}Y(\Lambda_{\chi_\varphi}\omega) + \frac{1}{n}Y(\theta_X(\chi_0) + \operatorname{Re} X(\varphi)) \\ &= \tilde{\Delta}Y(\varphi) + \frac{1}{n}(\operatorname{Re} X)(Y(\varphi)) + \frac{1}{n}Y(\theta_X(\chi_0)),\end{aligned}$$

where the last equality follows from the fact that the real part and imaginary part of a holomorphic vector field always commute.³

Claim: We always have $Y(\theta_X(\chi_0)) = 0$, thus $Y(\varphi)$ satisfies a very good parabolic equation and we conclude from maximum principle that $Y(\varphi) = 0$ along the flow.

To prove the claim, just note that from the definition of $\theta_X(\chi_0)$, we always have

$$X^i \chi_{i\bar{j}} = \partial_{\bar{j}} \theta_X(\chi_0).$$

So we have

$$X(\theta_X(\chi_0)) = X^j \bar{X}^i \chi_{j\bar{i}} = |X|_{\chi_0}^2.$$

Since both $X(\theta_X(\chi_0))$ and $\theta_X(\chi_0)$ are real, we have $(\operatorname{Im} X)(\theta_X(\chi_0)) = \operatorname{Im}(X(\theta_X(\chi_0))) = 0$. \square

From the above lemma, we see that actually we can rewrite our equation as

$$\begin{aligned}\frac{\partial \varphi}{\partial t} &= c - \frac{\omega \wedge \chi_\varphi^{n-1}}{\chi_\varphi^n} + \frac{1}{n} \theta_X(\chi_\varphi) \\ &= \frac{1}{n} \left(nc + \theta_X(\chi_\varphi) - \Lambda_{\chi_\varphi} \omega \right).\end{aligned}\tag{3.6}$$

Differentiating (3.6) with respect to t , we have

$$\frac{\partial}{\partial t} \frac{\partial \varphi}{\partial t} = \tilde{\Delta} \frac{\partial \varphi}{\partial t} + \frac{1}{n} X \left(\frac{\partial \varphi}{\partial t} \right).$$

The maximum principle implies that

$$\min_M \frac{\partial \varphi}{\partial t} \Big|_{t=0} \leq \frac{\partial \varphi}{\partial t} \leq \max_M \frac{\partial \varphi}{\partial t} \Big|_{t=0}.$$

In particular,

$$\Lambda_\chi \omega \leq \max_M \Lambda_{\chi_0} \omega + \max_M \theta_X(\chi_\varphi) - \min_M \theta_X(\chi_0).$$

Since both χ_0 and χ_φ are $\operatorname{Im} X$ -invariant by Zhou-Zhu [17], the term $\max_M \theta_X(\chi_\varphi) - \min_M \theta_X(\chi_0)$ is uniformly bounded. Thus, $\Lambda_\chi \omega$ has uniform positive upper bound along the flow. In particular, there is a uniform constant $c > 0$ such that

$$\chi_\varphi \geq c \omega \tag{3.7}$$

as long as the flow exists.

Lemma 3.5. *There is a uniform constant $C > 0$ such that for any (x, t) we have*

$$\Lambda_\omega \chi \leq C e^{A(\varphi - \inf_{M \times [0, t]} \varphi)},$$

and $|\varphi|_{C^0} \leq C$ as long as the flow exists.

³Note that a holomorphic vector field is always of the form $Z - iJZ$, where Z is real-holomorphic, i.e. $L_Z J = 0$. So we have $[Z, JZ] = L_Z(JZ) = JL_Z Z = 0$.

Proof. Following Song-Weinkove [12], in normal coordinates of ω , we have

$$\tilde{\Delta}(\Lambda_\omega \chi) = \frac{1}{n} h^{k\bar{l}} R_{k\bar{l}}{}^{i\bar{j}}(g) \chi_{i\bar{j}} + \frac{1}{n} h^{k\bar{l}} g^{i\bar{j}} \partial_k \partial_{\bar{l}} \chi_{i\bar{j}}, \quad (3.8)$$

where $R_{k\bar{l}}{}^{i\bar{j}}(g)$ denotes the curvature tensor of g . By the equation of the modified J -flow, we have

$$\begin{aligned} \frac{\partial}{\partial t} \Lambda_\omega \chi &= -\frac{1}{n} g^{i\bar{j}} \partial_i \partial_{\bar{j}} (\chi^{k\bar{l}} g_{k\bar{l}}) + \frac{1}{n} g^{i\bar{j}} \partial_i \partial_{\bar{j}} (\theta_X(\chi_\varphi)) \\ &= \frac{1}{n} \left(g^{i\bar{j}} h^{p\bar{q}} \partial_i \partial_{\bar{j}} \chi_{p\bar{q}} - g^{i\bar{j}} h^{r\bar{q}} \chi^{p\bar{s}} \partial_i \chi_{r\bar{s}} \partial_{\bar{j}} \chi_{p\bar{q}} - g^{i\bar{j}} h^{p\bar{s}} \chi^{r\bar{q}} \partial_i \chi_{r\bar{s}} \partial_{\bar{j}} \chi_{p\bar{q}} + \chi^{k\bar{l}} R_{k\bar{l}}(g) \right) \\ &\quad + \frac{1}{n} g^{i\bar{j}} \partial_i \partial_{\bar{j}} (\theta_X(\chi_\varphi)). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &(\tilde{\Delta} - \frac{\partial}{\partial t}) \log(\Lambda_\omega \chi) \\ &= \frac{\tilde{\Delta}(\Lambda_\omega \chi)}{\Lambda_\omega \chi} - \frac{|\tilde{\nabla}(\Lambda_\omega \chi)|^2}{(\Lambda_\omega \chi)^2} - \frac{\partial}{\partial t} \log(\Lambda_\omega \chi) \\ &= \frac{1}{n \Lambda_\omega \chi} \left(h^{k\bar{l}} R_{k\bar{l}}{}^{i\bar{j}}(g) \chi_{i\bar{j}} + g^{i\bar{j}} h^{r\bar{q}} \chi^{p\bar{s}} \partial_i \chi_{r\bar{s}} \partial_{\bar{j}} \chi_{p\bar{q}} + g^{i\bar{j}} h^{p\bar{s}} \chi^{r\bar{q}} \partial_i \chi_{r\bar{s}} \partial_{\bar{j}} \chi_{p\bar{q}} \right. \\ &\quad \left. - \chi^{k\bar{l}} R_{k\bar{l}}(g) - n \frac{|\tilde{\nabla}(\Lambda_\omega \chi)|^2}{\Lambda_\omega \chi} - g^{i\bar{j}} \partial_i \partial_{\bar{j}} \theta_X(\chi_\varphi) \right) \\ &\geq \frac{1}{n \Lambda_\omega \chi} \left(h^{k\bar{l}} R_{k\bar{l}}{}^{i\bar{j}}(g) \chi_{i\bar{j}} - \chi^{k\bar{l}} R_{k\bar{l}}(g) - g^{i\bar{j}} \partial_i \partial_{\bar{j}} \theta_X(\chi_\varphi) \right), \end{aligned}$$

where we used the inequality by Lemma 3.2 in [15]

$$n |\tilde{\nabla}(\Lambda_\omega \chi)|^2 \leq (\Lambda_\omega \chi) g^{i\bar{j}} h^{r\bar{q}} \chi^{p\bar{s}} \partial_i \chi_{r\bar{s}} \partial_{\bar{j}} \chi_{p\bar{q}}. \quad (3.9)$$

On the other hand, we have

$$\begin{aligned} X(\log \Lambda_\omega \chi) &= \frac{1}{\Lambda_\omega \chi} X \left(g^{i\bar{j}} (\chi_{0,i\bar{j}} + \varphi_{i\bar{j}}) \right) \\ &= \frac{1}{\Lambda_\omega \chi} \left(X(g^{i\bar{j}} \chi_{0,i\bar{j}}) + X(g^{i\bar{j}} \varphi_{i\bar{j}}) \right) \\ &= \frac{1}{\Lambda_\omega \chi} \left(X(g^{i\bar{j}} \chi_{0,i\bar{j}}) + \Delta_g(X(\varphi)) - X_{,i}^k \varphi_{k\bar{i}} \right), \end{aligned}$$

where we used the fact that

$$X(g^{i\bar{j}} \varphi_{i\bar{j}}) = X^k \varphi_{i\bar{i}k} = X^k \varphi_{k\bar{i}i} = g^{i\bar{j}} (X(\varphi))_{i\bar{j}} - X_{,i}^k \varphi_{k\bar{i}}.$$

Combining the above identities, we have

$$\begin{aligned}
& \left(\tilde{\Delta} + \frac{1}{n}X - \frac{\partial}{\partial t} \right) \log(\Lambda_\omega \chi) \\
& \geq \frac{1}{n\Lambda_\omega \chi} \left(h^{k\bar{l}} R_{k\bar{l}}^{i\bar{j}}(g) \chi_{i\bar{j}} - \chi^{k\bar{l}} R_{k\bar{l}}(g) - g^{i\bar{j}} \partial_i \partial_{\bar{j}} \theta_X(\varphi) + X(g^{i\bar{j}} \chi_{0,i\bar{j}}) + \Delta_g(X(\varphi)) - X_{,i}^k \varphi_{k\bar{i}} \right) \\
& = \frac{1}{n\Lambda_\omega \chi} \left(h^{k\bar{l}} R_{k\bar{l}}^{i\bar{j}}(g) \chi_{i\bar{j}} - \chi^{k\bar{l}} R_{k\bar{l}}(g) - X_{,i}^k \varphi_{k\bar{i}} - \Delta_g \theta_X + X(g^{i\bar{j}} \chi_{0,i\bar{j}}) \right) \\
& = \frac{1}{n\Lambda_\omega \chi} \left(h^{k\bar{l}} R_{k\bar{l}}^{i\bar{j}}(g) \chi_{i\bar{j}} - \chi^{k\bar{l}} R_{k\bar{l}}(g) - X_{,i}^k \chi_{k\bar{i}} + X_{,i}^k \chi_{0,k\bar{i}} - \Delta_g \theta_X + X(g^{i\bar{j}} \chi_{0,i\bar{j}}) \right),
\end{aligned}$$

where θ_X is the holomorphic potential of X with respect to χ_0 . Note that

$$\begin{aligned}
\left(\tilde{\Delta} + \frac{1}{n}X - \frac{\partial}{\partial t} \right) \varphi &= \frac{1}{n} \left(h^{k\bar{l}} \varphi_{k\bar{l}} + \chi^{i\bar{j}} g_{i\bar{j}} - nc - \theta_X \right) \\
&= \frac{1}{n} \left(2\chi^{i\bar{j}} g_{i\bar{j}} - h^{i\bar{j}} \chi_{0,i\bar{j}} - nc - \theta_X \right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
& n \left(\tilde{\Delta} + \frac{1}{n}X - \frac{\partial}{\partial t} \right) \left(\log(\Lambda_\omega \chi) - A\varphi \right) \\
& \geq \frac{1}{\Lambda_\omega \chi} \left(h^{k\bar{l}} R_{k\bar{l}}^{i\bar{j}}(g) \chi_{i\bar{j}} - \chi^{k\bar{l}} R_{k\bar{l}}(g) - X_{,i}^k \chi_{k\bar{i}} + C(\chi_0, \omega, X) \right) \\
& \quad - 2A\chi^{i\bar{j}} g_{i\bar{j}} + Ah^{i\bar{j}} \chi_{0,i\bar{j}} + ncA + A\theta_X.
\end{aligned}$$

By the assumption (3.4), we can choose $\epsilon > 0$ sufficiently small such that

$$(nc\chi_0 - (n-1)\omega) \wedge \chi_0^{n-2} + \theta_X(\chi_0) \chi_0^{n-1} > 2\epsilon \chi_0^{n-1}. \quad (3.10)$$

Moreover, since χ_φ is uniformly bounded from below, we can choose A large such that

$$-\frac{1}{A\Lambda_\omega \chi} \left(h^{k\bar{l}} R_{k\bar{l}}^{i\bar{j}}(g) \chi_{i\bar{j}} - \chi^{k\bar{l}} R_{k\bar{l}}(g) - X_{,i}^k \chi_{k\bar{i}} + C(\omega, X) \right) \leq \epsilon,$$

then at the maximum point (x_0, t_0) of $\log(\Lambda_\omega \chi) - A\varphi$, we have

$$nc + \theta_X + h^{i\bar{j}} \chi_{0,i\bar{j}} - 2\chi^{i\bar{j}} g_{i\bar{j}} \leq \epsilon.$$

We choose normal coordinates for the metric χ_0 so that the metric χ is diagonal with entries $\lambda_1, \dots, \lambda_n$. We denote the diagonal entries of ω by μ_1, \dots, μ_n . Thus, we have

$$nc + \theta_X(x_0) + \sum_{i=1}^n \frac{\mu_i}{\lambda_i^2} - 2 \sum_{i=1}^n \frac{\mu_i}{\lambda_i} \leq \epsilon,$$

which implies that for any fixed index k , we have the inequality

$$\begin{aligned}
\epsilon & \geq \sum_{i=1, i \neq k}^n \mu_i \left(\frac{1}{\lambda_i} - 1 \right)^2 - \sum_{i=1, i \neq k}^n \mu_i + \frac{\mu_k}{\lambda_k^2} - 2 \frac{\mu_k}{\lambda_k} + nc + \theta_X(x_0) \\
& \geq nc + \theta_X(x_0) - \sum_{i=1, i \neq k}^n \mu_i - 2 \frac{\mu_k}{\lambda_k}.
\end{aligned} \quad (3.11)$$

On the other hand, by (3.10) we have for any k ,

$$(nc\chi_0 - (n-1)\omega) \wedge \chi_0^{n-2} \wedge \beta_k + \theta_X(\chi_0) \chi_0^{n-1} \wedge \beta_k > 2\epsilon \chi_0^{n-1} \wedge \beta_k,$$

where $\beta_k := \sqrt{-1}dz^k \wedge d\bar{z}^k$. This means

$$nc + \theta_X(x_0) - \sum_{i=1, i \neq k}^n \mu_i > 2\epsilon.$$

Combining the above inequalities, we have $\frac{\lambda_k}{\mu_k} < \frac{2}{\epsilon}$ and there is a constant $C = C(n, \epsilon)$ such that at the point (x_0, t_0) ,

$$\Lambda_\omega \chi \leq C.$$

Thus, at any point (x_0, t_0) we have the estimate

$$\Lambda_\omega \chi \leq Ce^{A(\varphi - \inf_{M \times [0, t]} \varphi)}.$$

The passage from this C^2 estimate to C^0 estimate does not use the equation and hence is identical to Song-Weinkove[11] and Weinkove [15][16], so we omit it. □

Corollary 3.6. *The modified J -flow exists for any $t \in [0, \infty)$.*

Proof. Suppose the solution exists only in $[0, T)$ with $T < \infty$. We will derive a contradiction. By the above lemma, we have uniform C^0 and C^2 estimates. By interpolation, we also have uniform C^1 estimate on $[0, T)$. By Evans-Krylov estimate, we also have uniform $C^{2, \alpha}$ estimate. Then we can take limit of $\varphi(\cdot, t_i)$ as $t_i \rightarrow T$ to get a φ_T . Since χ_φ is uniformly bounded from below, $\chi_0 + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\varphi_T$ is a Kähler form. So the solution can extend beyond T , a contradiction! □

Now we can use the modified J -flow to finish the proof of Theorem 3.3:

Proof of Theorem 3.3. By our above discussion, the modified J -flow has a unique solution $\varphi(\cdot, t)$ for $t \in [0, \infty)$. By the proof of the above theorem, we also have a uniform $C^{2, \alpha}$ estimate. By the standard bootstrap argument, the solutions are uniformly bounded with respect to any C^k norm. Then for any sequence $t_i \rightarrow \infty$, we can find a subsequence, also denoted by t_i such that $\varphi(\cdot, t_i) \rightarrow \varphi_\infty$ in C^∞ . We shall prove that φ_∞ solves (3.3).

To show this, we define an energy functional associated with the modified J -functional following Chen [2]:

$$E_{X, \chi_0}(\varphi) := \int_M (\theta_X(\chi_\varphi) - \Lambda_{\chi_\varphi} \omega)^2 \frac{\chi_\varphi^n}{n!} = \int_M \sigma^2 \frac{\chi_\varphi^n}{n!},$$

where $\sigma := \theta_X(\chi_\varphi) - \Lambda_{\chi_\varphi}\omega$. Then along the modified J-flow, we have (in normal coordinates of χ_φ)

$$\begin{aligned}
\frac{d}{dt}E_{X,\chi_0}(\varphi) &= \int_M \left[2\sigma \left(X \left(\frac{\partial \varphi}{\partial t} \right) + \chi_\varphi^{i\bar{q}} \chi_\varphi^{p\bar{j}} \left(\frac{\partial \varphi}{\partial t} \right)_{,p\bar{q}} g_{i\bar{j}} \right) + \sigma^2 \Delta_\chi \frac{\partial \varphi}{\partial t} \right] \frac{\chi_\varphi^n}{n!} \\
&= \int_M \left[\frac{2}{n} \sigma X(\sigma) + \frac{2}{n} \sigma \chi_\varphi^{i\bar{q}} \chi_\varphi^{p\bar{j}} \sigma_{,p\bar{q}} g_{i\bar{j}} + \frac{1}{n} \sigma^2 \chi_\varphi^{p\bar{q}} \sigma_{,p\bar{q}} \right] \frac{\chi_\varphi^n}{n!} \\
&= \int_M \left[\frac{2}{n} \sigma X(\sigma) - \frac{2}{n} \chi_\varphi^{i\bar{q}} \chi_\varphi^{p\bar{j}} \sigma_{,\bar{q}} \sigma_{,p} g_{i\bar{j}} - \frac{2}{n} \sigma \chi_\varphi^{i\bar{q}} \chi_\varphi^{p\bar{j}} \sigma_{,p} g_{i\bar{j},\bar{q}} - \frac{2}{n} \sigma \chi_\varphi^{p\bar{q}} \sigma_{,\bar{q}} \sigma_{,p} \right] \frac{\chi_\varphi^n}{n!} \\
&= \int_M \left[\frac{2}{n} \sigma X(\sigma) - \frac{2}{n} \chi_\varphi^{i\bar{q}} \chi_\varphi^{p\bar{j}} \sigma_{,\bar{q}} \sigma_{,p} g_{i\bar{j}} - \frac{2}{n} \sigma \chi_\varphi^{i\bar{q}} \chi_\varphi^{p\bar{j}} \sigma_{,p} g_{i\bar{q},\bar{j}} - \frac{2}{n} \sigma \chi_\varphi^{p\bar{q}} \sigma_{,\bar{q}} \sigma_{,p} \right] \frac{\chi_\varphi^n}{n!} \\
&= \int_M \left[\frac{2}{n} \sigma X(\sigma) - \frac{2}{n} \chi_\varphi^{i\bar{q}} \chi_\varphi^{p\bar{j}} \sigma_{,\bar{q}} \sigma_{,p} g_{i\bar{j}} - \frac{2}{n} \sigma \chi_\varphi^{p\bar{j}} \sigma_{,p} (\Lambda_{\chi_\varphi} \omega)_{,\bar{j}} - \frac{2}{n} \sigma \chi_\varphi^{p\bar{q}} \sigma_{,\bar{q}} \sigma_{,p} \right] \frac{\chi_\varphi^n}{n!} \\
&= \int_M \left[\frac{2}{n} \sigma X(\sigma) - \frac{2}{n} \chi_\varphi^{i\bar{q}} \chi_\varphi^{p\bar{j}} \sigma_{,\bar{q}} \sigma_{,p} g_{i\bar{j}} - \frac{2}{n} \sigma \chi_\varphi^{p\bar{q}} (\theta_X(\chi_\varphi))_{,\bar{q}} \sigma_{,p} \right] \frac{\chi_\varphi^n}{n!} \\
&= -\frac{2}{n} \int_M \chi_\varphi^{i\bar{q}} \chi_\varphi^{p\bar{j}} \sigma_{,\bar{q}} \sigma_{,p} g_{i\bar{j}} \frac{\chi_\varphi^n}{n!}.
\end{aligned}$$

The last equality comes from the definition of $\theta_X(\chi_\varphi)$. So we have

$$\frac{d}{dt}E_{X,\chi_0}(\varphi) = -\frac{2}{n} \int_M |\nabla^{\chi} \sigma|_\omega^2 \frac{\chi_\varphi^n}{n!} < 0.$$

In particular, this implies

$$\int_0^\infty \left(\int_M |\nabla^{\chi_t} \sigma(\cdot, t)|_\omega^2 \frac{\chi_t^n}{n!} \right) dt < \infty.$$

So if the sequence t_i is chosen properly, so that

$$\int_M |\nabla^{\chi_{t_i}} \sigma(\cdot, t_i)|_\omega^2 \frac{\chi_{t_i}^n}{n!} \rightarrow 0,$$

then we can conclude that

$$\theta_X(\chi_{\varphi_\infty}) - \Lambda_{\chi_{\varphi_\infty}} \omega \equiv \text{const.}$$

This implies that φ_∞ solves (3.3). The theorem is proved. \square

3.3. Proof of Theorem 1.2.

Proof of Theorem 1.2. We focus on the $G = \{1\}$ case, the proof in the general case is identical. Recall the Aubin-Yau functionals

$$\begin{aligned}
I_{\chi_0}(\varphi) &= \int_X \varphi \left(\frac{\chi_0^n}{n!} - \frac{\chi_\varphi^n}{n!} \right), \\
J_{\chi_0}(\varphi) &= \int_0^1 dt \int_X \frac{\partial \varphi_t}{\partial t} \left(\frac{\chi_0^n}{n!} - \frac{\chi_\varphi^n}{n!} \right).
\end{aligned}$$

Direct calculation shows that

$$\begin{aligned} I_{\chi_0}(\varphi) - J_{\chi_0}(\varphi) &= - \int_0^1 dt \int_X \frac{\partial \varphi}{\partial t} \Delta_{\chi_\varphi} \varphi \frac{\chi_\varphi^n}{n!} \\ &= - \int_0^1 \int_X \frac{\partial \varphi}{\partial t} (\chi_\varphi^n - \chi_0 \wedge \chi_\varphi^{n-1}) \frac{dt}{(n-1)!}. \end{aligned}$$

As \hat{J} , we also have the following lemma for \tilde{J} , whose proof is the same as in [13] and [9]:

Lemma 3.7. *If $\tilde{J}_{\omega, \chi}$ is bounded from below, then so is $\tilde{J}_{\omega', \chi}$ for any $\omega' \in [\omega]$. (ω' may not be positive.)*

Set

$$c := \frac{(\epsilon[\chi_0] - \pi c_1(M)) \cdot [\chi_0]^{n-1}}{[\chi_0]^n} = - \frac{\pi c_1(M) \cdot [\chi_0]^{n-1}}{[\chi_0]^n} + \epsilon.$$

By condition (2) of Theorem 1.2, we can find a $\omega \in \epsilon[\chi_0] - \pi c_1(M)$ with $\omega > 0$ (Since $\min \theta_X < 0$, $\epsilon[\chi_0] - \pi c_1(M) > 0$). Since $\text{Im} X$ generates a compact one-parameter group of holomorphic automorphisms, we can average ω . So we can also assume that $L_{\text{Im} X} \omega = 0$. Then by condition (3), we can find a closed (1,1)-form $\chi' \in [\chi_0]$ such that

$$(nc + \min \theta_X) \chi' - (n-1) \omega > 0.$$

We claim that $nc + \min \theta_X > 0$, thus the above inequality also implies $\chi' > 0$. In fact, by condition (2), we have

$$\frac{\pi c_1(M) \cdot [\chi_0]^{n-1}}{[\chi_0]^n} < \epsilon + \min \theta_X.$$

So, $c > -\min \theta_X$, and hence

$$nc + \min \theta_X > -(n-1) \min \theta_X > 0.$$

Now we have

$$(nc + \theta_X(\chi')) \chi' - (n-1) \omega \geq (nc + \min \theta_X) \chi' - (n-1) \omega > 0.$$

By Theorem 3.3, we know that $\tilde{J}_{\omega, \chi_0}$ has critical point. By Lemma 3.7 and 3.2, we conclude that (Remember that $\omega_0 = -\text{Ric}(\chi_0)$)

$$\tilde{J}_{\omega_0 + \epsilon \chi_0, \chi_0} \geq -C.$$

By (3.2), we have

$$\begin{aligned} \tilde{\mu}_{\chi_0}(\varphi) &= \int_M \log \frac{\chi_\varphi^n}{\chi_0^n} \frac{\chi_\varphi^n}{n!} + \tilde{J}_{\omega_0, \chi_0}(\varphi) \\ &= \int_M \log \frac{\chi_\varphi^n}{\chi_0^n} \frac{\chi_\varphi^n}{n!} + \tilde{J}_{\omega_0 + \epsilon \chi_0, \chi_0}(\varphi) - \epsilon(I_{\chi_0} - J_{\chi_0})(\varphi) \\ &\geq \left(\frac{n+1}{n} \alpha - \epsilon \right) (I_{\chi_0} - J_{\chi_0})(\varphi) - C, \end{aligned}$$

for any positive $\alpha < \alpha_M([\chi_0])$. By condition (1), we can choose such an $\alpha < \alpha_M([\chi_0])$ with $\frac{n+1}{n} \alpha - \epsilon > 0$, so the modified K -energy is proper. \square

REFERENCES

- [1] Chen, X. X. *On the lower bound of the Mabuchi energy and its application*. Internat. Math. Res. Notices 2000, no. 12, 607–623.
- [2] Chen, X. X. *A new parabolic flow in Kähler manifolds*. Comm. Anal. and Geom. 2004, no. 12, 837–852.
- [3] Chen, X. X. and Tian, G. *Geometry of Kähler metrics and foliations by holomorphic discs*. Publ. Math. Inst. Hautes études Sci. No. 107 (2008), 1–107.
- [4] Dervan, R., *Alpha invariants and coercivity of the Mabuchi functional on Fano manifolds*. arXiv:1412.1426.
- [5] Donaldson, S. K. *Moment maps and diffeomorphisms*. Asian J. Math. 3 (1999), no. 1, 1–15.
- [6] Fang, H., Lai, X., Song, J., Weinkove, B. *The J-flow on Kähler surfaces: a boundary case*. arXiv:1204.4068.
- [7] Fulton, W. *Introduction to toric varieties*. Princeton University Press, 1993.
- [8] Futaki, A., Mabuchi, T. *Bilinear forms and extremal Kähler vector fields associated with Kähler classes*. Math. Ann. 301 (1995), no.2, 199–210.
- [9] Li, H., Shi, Y., Yao, Y., *A criterion for the properness of the K-energy in a general Kähler class*. Math. Ann. (published on line 2014: DOI 10.1007/s00208-014-1073-z).
- [10] Mabuchi, T., *A Theorem of Calabi-Matsushima’s Type*. Osaka J. Math. 39 (2002), 49–57.
- [11] Song, J., Weinkove, B. *On the convergence and singularities of the J-flow with applications to the Mabuchi energy*. Comm. Pure Appl. Math. 61 (2008), no. 2, 210–229.
- [12] Song, J., Weinkove, B. *The degenerate J-flow and the Mabuchi energy on minimal surfaces of general type*. arXiv:1309.3810.
- [13] Székelyhidi, G. *Greatest lower bounds on the Ricci curvature of Fano manifolds*. Compositio Math. 147 (2011), 319–331.
- [14] Tian, G., *Canonical metrics in Kähler geometry*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2000. Notes taken by Meike Akveld.
- [15] Weinkove, B. *Convergence of the J-flow on Kähler surfaces*. Comm. Anal. Geom. 12 (2004), no. 4, 949–965.
- [16] Weinkove, B. *On the J-flow in higher dimensions and the lower boundedness of the Mabuchi energy*. J. Differential Geom. 73 (2006), no. 2, 351–358.
- [17] Zhou, B. and Zhu, X. *Relative K-stability and modified K-energy on toric manifolds*. Advances in Math. 219 (2008), 1327–1362.

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